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CALCULATION OF THE BOUNDARY DERIVATIVES OF THE SOLUTIONS OF THE FIRST AND SECOND BOUNDARY-VALUE PROBLEMS FOR POISSON'S EQUATION†

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Expressions are obtained for calculating the normal derivative of the solution of the Dirichlet problem and the tangential derivative of the solution of the Neumann problem for Poisson's equation in terms of the parameters of the one-to-one and conformal mapping of the plane region into the unit circle.

1. FUNDAMENTAL EQUATIONS

Suppose u is the solution of Poisson's equation $\Delta u=-1$ in the plane region D, which on the boundary Γ of this region satisfies either the Dirichlet conditions $u|_\Gamma=0$ (problem 1), or the Neumann conditions $\partial u/\partial n|_\Gamma=A$ (problem 2), where $\partial/\partial n$ is the derivative with respect to the output normal to Γ.

We know that the solution of problem 2 can be found, apart from a constant, while the constant $A=|D|/|\Gamma|$, where $|D|$ is the area of the region and $|\Gamma|$ is the perimeter of the boundary of Γ.

The purpose of the present paper is to obtain expressions for calculating the normal derivative of the solution of problem 1 and the tangential derivative of the solution of problem 2 on the boundary.

We know that the solution of problem 1 is proportional to the stress function in the problem of the torsion of elastic prismatic rods, the cross-section of which is the region D, while the normal derivative on the boundary is proportional to the shear stress [1]. Estimates of u_n are given in a number of papers ([2, 3], etc.). However, it is not only the values of the function $u_n(s)$ (s is the natural parameter on Γ) that are important, but also the arrangement of the maxima of this function (the dangerous points). Saint Venant obtained analytic solutions of problem 1 for a large number of regions (an ellipse, a right triangle, etc.). For all these regions the dangerous points are situated at the points Γ that are least distant from the centre of symmetry of the region. It was shown in [4] that with certain additional assumptions regarding the boundary Γ, u_n reaches extrema only at points lying on the axes of symmetry of the region D (to be sure, if there are two such axes). Below we obtain an expression for u_n in terms of the coefficients of the conformal mapping of the region D into the unit circle, which enables the list of regions for which dangerous points can be found explicitly to be extended.

Problem 2 is the linearized static problem of the form of the free surface of a liquid in a cylindrical capillary, the cross-section of which is the region D, due to the action of the surface-tension forces. Knowing the derivative of $u(s)$ on Γ we can obtain complete information on the

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oscillations of the solution itself on the boundary and, in particular, obtain the points Γ at which the liquid rises to the greatest height.

We will first obtain expressions which we will devote the rest of the paper to investigating. We will use Green's formula

$$\iint_D (\Delta f \cdot g - f\Delta g)dxdy = \int_\Gamma \left(\frac{\partial f}{\partial n} g - f\frac{\partial g}{\partial n}\right)ds \tag{1.1}$$

Theorem 1. Suppose $u(x, y)$ is the solution of problem 1. Suppose further that $\{\upsilon_{s,h}\}$ is a family of functions, harmonic in the region D, possessing the following properties: $\upsilon_{s,h} > 0$ in the region D, $\upsilon_{s,h} > 0$ on the arc $\Gamma \setminus (s-h,\ s+h)$ and $\int_\Gamma \upsilon_{s,h} ds = 1$.

Then

$$u_n(s) = \lim_{h\to 0} I_h(s_0) \tag{1.2}$$

where

$$I_h(s_0) = \iint_D \upsilon_{s,h} dxdy$$

Proof. Putting $f = u$ and $g = \upsilon_{s,h}$ in (1.1), and using the formulation of problem 1, we obtain

$$I_h(s_0) = \int_\Gamma u_n \upsilon_{s,h} ds \tag{1.3}$$

Now applying the theorem of the mean to the contour integral and using the properties of the family $\{\upsilon_{s,h}\}$, we can conclude that it is equal to $u_n(\tilde{s})(\tilde{s} \in (s-h,\ s+h))$.

Passing to the limit as $h \to 0$ in (1.3), we obtain (1.2).

Theorem 2. Suppose $u(x, y)$ is the solution of problem 2. Suppose further that $\{\upsilon'_{s,h}\}$ is a family of functions, harmonic in the region D, such that the functions of the family $\{\upsilon_{s,h}\}$ are conjugate to them. Then

$$u(s) = \lim_{h\to 0}\left(\frac{|D|}{|\Gamma|} E_h(s_0) + J_h(s_0)\right) \tag{1.4}$$

where

$$E_h(s_0) = \int_\Gamma \upsilon'_{s,h} ds, \quad J_h(s_0) = -\iint_D \upsilon'_{s,h} dxdy$$

(the meaning of the argument s_0 will become clear in the next section).

Proof. Putting $f = u$ and $g = \upsilon'_{s,h}$ in (1.1) and using the formulation of problem 2, we obtain

$$-J_h(s_0) = \frac{|D|}{|\Gamma|} E_h(s_0) - \int_\Gamma u \frac{\partial \upsilon'_{s,h}}{\partial n} ds \tag{1.5}$$

Converting the last integral in (1.5), using the Cauchy–Riemann conditions and integrating by parts, Eq. (1.5) takes the form

$$\int_\Gamma u \upsilon_{s,h} ds = \frac{|D|}{|\Gamma|} E_h(s_0) + J_h(s_0) \tag{1.6}$$

Further, as in the proof of Theorem 1, we use the properties of the family $\{\upsilon_{s,h}\}$ and pass to the limit in (1.6) as $h \to 0$. We obtain Eq. (1.4).

2. CALCULATION OF THE NORMAL DERIVATIVE IN PROBLEM 1 USING SERIES

Harmonic functions occur in (1.2) and (1.4), and hence it is natural to calculate the boundary derivatives using series which realize a conformal mapping of the region into the unit circle. We will show how such calculations can be carried out in problem 1.

Suppose $z = x + iy$ and $\zeta = \xi + i\eta = re^{i\varphi}$. Suppose the function $z = f(\zeta)$ carries out a one-to-one conformal mapping of region D into the unit circle $B(|\zeta| < 1)$

$$z = a_1\zeta + a_2\zeta^2 + \dots + a_n\zeta^n + \dots \equiv f(\zeta) \tag{2.1}$$

Everywhere henceforth we will assume that the following numerical series converges

$$|a_1| + 2|a_2| + \dots + n|a_n| + \dots < +\infty \tag{2.2}$$

We will take as the elements of the family $\{\upsilon_{s,h}\}$ the solutions of the Dirichlet problem with the following boundary conditions: $\upsilon_{s,h} = 1/(2h)$ on the arc $(s-h,\ s+h)$ and $\upsilon_{s,h} = 0$ on the remaining part of Γ. Note that the functions $w(\zeta) = \upsilon_{s,h}(f(\zeta))$ and $w^*(\zeta) = \upsilon'_{s,h}(f(\zeta))$ are harmonic in the circle B, and the function $w(e^{i\varphi})$ is equal to zero outside the arc of the circle $(s_0 - \varepsilon,\ s_0 + \varepsilon)$ and equal to $1/(2h)$ on this arc. (The arc $(s_0 - \varepsilon,\ s_0 + \varepsilon)$ is the inverse image of the arc $(s-h,\ s+h)$ in mapping (2.1).)

It follows from elementary calculations that

$$w(r,\varphi) = \frac{\varepsilon}{2\pi h} + \frac{1}{2\pi h}\sum_{k=1}^{\infty} \frac{r^k \sin k\varepsilon}{k} \cos k(\varphi - s_0) \tag{2.3}$$

$$w^*(r,\varphi) = -\frac{1}{\pi h}\sum_{k=1}^{\infty} \frac{r^k \sin k\varepsilon}{k} \sin k(\varphi - s_0) \tag{2.4}$$

Note also that it follows from the convergence of series (2.2) that the following limit exists

$$\lim_{h\to 0} \frac{\varepsilon}{h} = C(s_0) = |f'(e^{is_0})|^{-1} \tag{2.5}$$

Theorem 3. Suppose that $a_n = |a_n| e^{i\theta_n}$, and $\theta_{ln} = \theta_l - \theta_n$. in (2.1). Then

$$|u_n(s)| = \tfrac{1}{2}(S_1 + S_2)(S_3 + S_4)^{-1/2} \tag{2.6}$$

where

$$S_1 = \sum_{l=1}^{\infty} l|a_l|^2; \quad S_3 = \sum_{l=1}^{\infty} l^2|a_l|^2$$

$$S_2 = 2\sum_{l=1}^{\infty} \sum_{n<l} n|a_n||a_l|\cos(\theta_{ln} + (l-n)s_0)$$

$$S_4 = 2\sum_{l=1}^{\infty} \sum_{n<l} nl|a_n||a_l|\cos(\theta_{ln} + (l-n)s_0)$$

Proof. It follows from Theorem 1 that it is sufficient to evaluate the integral

$$I_h(s_0) = \iint_B w(\zeta)|f'(\zeta)|^2\, d\xi d\eta \tag{2.7}$$

and then pass to the limit as $h \to 0$.

We will use the well-known method from [6] in the calculation. From (2.1) we obtain

$$|f'(\zeta)|^2 = f'(\zeta)\overline{f'(\zeta)} = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} nla_n a_l \zeta^{n-1}\bar{\zeta}^{l-1} \tag{2.8}$$

We substitute (2.3) and (2.8) into (2.7) and then use the equalities

$$\cos k(\varphi - s_0) = \tfrac{1}{2}(e^{ik(\varphi - s_0)} + e^{-ik(\varphi - s_0)}); \quad a_n = |a_n|e^{i\theta_n}$$

and the conditions of orthogonality. Then

$$I_h(s_0) = \frac{\varepsilon}{2\pi h}\sum_{l=1}^{\infty}\int_0^{2\pi}\int_0^1 a_l\bar{a}_l r^{2l-1} l^2 dr d\varphi +$$

$$+\frac{1}{2\pi h}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^1 \frac{\sin k\varepsilon}{2k}(e^{ik(\varphi - s_0)} + e^{-ik(\varphi - s_0)})\times$$

$$\times nl|a_n||a_l| r^{n+l+k-1} e^{i\theta_{nl}} e^{i(n-l)} dr d\varphi =$$

$$= \frac{\varepsilon}{2h}S_1 + \frac{1}{\pi h}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{2\pi}\int_0^1 nl|a_n||a_l|\times$$

$$\times\frac{\sin k\varepsilon}{2k} e^{i\theta_{nl}} r^{n+l+k-1}(e^{-iks_0}e^{i(n-l+k)\varphi} + e^{iks_0}e^{i(n-l-k)\varphi}) dr d\varphi =$$

$$= \frac{\varepsilon}{2h}S_1 + \sum_{l=1}^{\infty}\sum_{n<l}\frac{n\sin(l-n)\varepsilon}{h(l-n)}|a_n||a_l|\cos(\theta_{ln} + (l-n)s_0) \tag{2.9}$$

We now pass to the limit in (2.9) as $h \to 0$. It follows from the convergence of series (2.2) that this limit exists and is equal to

$$I(s_0) = \lim_{h\to 0} I_h(s_0) = \frac{1}{2}C(s_0)(S_1 + S_2) \tag{2.10}$$

On the other hand, it follows from (2.5), (2.8) and Euler's formulae that

$$C(s_0) = |f'(e^{is_0})|^{-1} = (S_3 + S_4)^{-1/2} \tag{2.11}$$

Substituting (2.11) into (2.10) we obtain (2.6).

Formula (2.6) enables us to calculate the normal derivative of the solution of problem 1 at any point of the boundary Γ. However, it is not possible to solve the problem of the extrema of $u_n(s)$ completely using this formula. On the other hand, it is possible to obtain the position of these extrema in cases that differ from the classical ones.

Example. Suppose that

$$f(\zeta) = a_1\zeta + a_2\zeta^2$$

in (2.1). The condition for the function $f(\zeta)$ to be one-sheeted has the form $|a_1|/(2|a_2|) > 1$. For the quantities S_m in (2.6) we obtain

$$S_1 = |a_1|^2 + 2|a_2|^2, \quad S_2 = 2|a_1||a_2|\cos(\theta_{21} + s_0)$$
$$S_3 = |a_1|^2 + 4|a_2|^2, \quad S_4 = 2S_2$$

It is easy to obtain the extrema of the function $|u_n(s)|$. It reaches its maxima at points on the boundary Γ such that $\sin(\theta_{21}+s_0)=0$, and minima at points such that $\cos(\theta_{21}+s_0)=-|a_2|/|a_1|$. The greatest maximum is reached at the point where $\cos(\theta_{21}+s_0)=-1$ (a dangerous point). The greatest value of the modulus of the normal derivative is

$$|u_n|_{\max} = {}^1/_2(|a_1|^2 + 2|a_2|^2 - 2|a_1||a_2|)/(|a_1| - 2|a_2|)$$

while the least value is $|u_n|_{\min}=|a_1|/2$.

A formula similar to (2.6) can also be obtained for $u(s)$ in problem 2. It is merely necessary, when evaluating the double integral in (1.4), to use function (2.4), and when evaluating the contour integral one must assume the expansion of the function $|f(e^{i\varphi})|$ in a Fourier series to be known. We will not give this formula here in view of its complexity.

3. CALCULATION OF THE BOUNDARY DERIVATIVES OF PROBLEMS 1 AND 2 FOR ALMOST CIRCULAR REGIONS

Suppose now that the region D is almost the unit circle. By this we mean the following. We will assume that

$$|f'(\zeta)| = 1 + \alpha(r, \varphi), \quad \alpha(r, \varphi) \ll 1 \tag{3.1}$$

Our purpose is to obtain approximate expressions (accurate to α^2) for the boundary derivatives of problems 1 and 2.

In this formulation we must obtain an answer which does not contain infinite series. Thus, we will neglect expressions of the order of α^2 and higher everywhere below. We know that $\ln|f(\xi)|$ is an harmonic function in the circle B (since the derivative $f(\zeta)$ is not equal to zero in B). On the other hand, the following equality holds to within α^2

$$\ln|f'(\zeta)| = \alpha(r, \varphi)$$

i.e. the function α is harmonic in the circle B and can be expanded in a Fourier series

$$\alpha(r,\varphi) = \frac{a_0}{2} + \sum_{l=1}^{\infty} r^l(a_l\cos l\varphi + b_l \sin l\varphi) = \sum_{l=-\infty}^{\infty} A_l r^{|l|}e^{il\varphi} \tag{3.2}$$

The coefficients a_1, b_1 and A_1 are connected by well-known relations. Suppose $A_l=|A_l|e^{i\theta_l}$. Then

$$A_{-l} = \bar{A}_l, \quad a_l = 2|A_l|\cos\theta_l, \quad b_l = -2|A_l|\sin\theta_l \quad (l>0) \tag{3.3}$$

We will use the following notation

$$t_k = (a_k \cos ks_0 + b_k \sin ks_0)/(k+1), \quad \bar{t}_k = (a_k \sin ks_0 - b_k \cos ks_0)/(k+1)$$

The calculation of $u_n(s)$ in problem 1. As in Section 2, we will evaluate the integral (2.7). To do this we substitute expressions (2.3) and (3.2) into it, and we then transform, using the conditions of orthogonality and (3.3)). We obtain

$$I_h(s_0) = \frac{\varepsilon}{2h} + \frac{\varepsilon}{h}\frac{a_0}{2} + \frac{1}{h}\sum_{k=1}^{\infty}\frac{\sin k\varepsilon}{k}t_k$$

We pass to the limit in this equality as $h\to 0$, using (2.5). Summing the series we obtain

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} t_k=\int_0^1 \alpha(r,s_0)dr$$

and then, making use of (3.1) and Theorem 1, we obtain

$$|u_n(s)|=(1+\alpha(1,s_0))^{-1}\left(\frac{1}{2}+\int_0^1 \alpha(r,s_0)dr\right) \tag{3.4}$$

Calculation of $u(s)$ in problem 2. Using (1.4) we can evaluate the double and contour integrals. The double integral is evaluated in the same way as the previous one, except that we must use (2.4) instead of (2.3). Then

$$J_h(s_0)=-\frac{1}{h}\sum_{k=1}^{\infty}\frac{\sin k\varepsilon}{k}\bar{t}_k$$

We pass to the limit in this equality as $h\to 0$, using (2.5) and summing the series

$$\sum_{k=1}^{\infty}\bar{t}_k=-\int_0^1\left(\int_0^r \frac{1}{\rho}\frac{\partial\alpha}{\partial s_0}d\rho\right)dr=-\int_0^1 \bar{\alpha}(r,s_0)dr \tag{3.5}$$

Here $\bar{a}(r,\ \varphi)$ is a harmonic function, conjugate to $\alpha(r,\ \varphi)$ and such that $\bar{\alpha}(0,\ \varphi)=0$. We finally obtain

$$J(s_0)=-(1+\alpha(1,s_0))^{-1}\int_0^1 \alpha(r,s_0)dr \tag{3.6}$$

We will now evaluate the contour integral. Using (2.4), (3.2), (3.3) and the orthogonality conditions we obtain

$$E_h(s_0)=\frac{1}{h}\sum_{k=1}^{\infty}\frac{\sin k\varepsilon}{k}(k+1)\bar{t}_k$$

We pass to the limit in this equality as $h\to 0$, using (2.5), and we then use (3.5) and the Cauchy–Riemann conditions. We obtain

$$E(s_0)=\lim_{h\to 0}E_h(s_0)=-|f'(e^{is_0})|^{-1}\int_0^1\frac{1}{r}\frac{\partial\alpha}{\partial s_0}dr=|f'(e^{is_0})|^{-1}\bar{\alpha}(1,s_0) \tag{3.7}$$

Using (3.1) and Theorem 2 and combining (3.6) and (3.7), we obtain

$$u^{\cdot}(s)=(1+\alpha(1,s_0))^{-1}\left(\frac{|D|}{|\Gamma|}\bar{\alpha}(1,s_0)-\int_0^1 \bar{\alpha}(r,s_0)dr\right) \tag{3.8}$$

Example. Suppose the region D is an ellipse with semiaxes $b=1$ and $a=1+\varepsilon$. We know (see [7, p. 378]), that, to within ε^2, the mapping (2.1) has the form

$$z=\zeta+{}^1\!/\!{}_2(\zeta+\zeta^3)$$

Then, with the same accuracy, it follows from (3.1) that

$$\alpha(r,\varphi)=\tfrac{1}{2}\varepsilon+\tfrac{3}{2}\varepsilon r^2\cos 2\varphi,\quad \bar{\alpha}(r,\varphi)=\tfrac{3}{2}\varepsilon r^2\sin 2\varphi \tag{3.9}$$

Substituting the first expression of (3.9) into (3.4) we obtain the solution of problem 1

$$|u_n(s)| = 1/2 - 1/4\varepsilon(\cos 2s_0 - 1) + O(\varepsilon^2)$$

already known to Saint-Venant.

In order to obtain a solution of problem 2, we note that, to terms in ε^2, the ratio $|D|/l = \frac{1}{2} + \frac{1}{4}\varepsilon$. Substituting the second expression of (3.9) into (3.8) we obtain

$$u'(s) = 1/4\sin 2s_0 + O(\varepsilon^2)$$

Hence, both for the function $|u_n(s)|$, and for the function $u(s)$ the greatest values are reached at the ends of the minor axis of the ellipse, and the smallest values are reached at the ends of the major axis.

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